Degenerations in Gromov-Witten and Donaldson-Thomas Theories

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Motivation

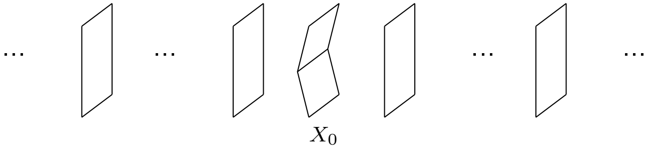
- Let X be a smooth projective scheme.
- Let M_X be the moduli space of some geometric objects on X with some numerical invariant i_X.
  - Stable maps and Gromov-Witten invariants.
  - Ideal sheaves and Donaldson-Thomas invariants.

Motivation (cont.)

- Let $X_0 = Y_1 \cup_D Y_2$ be a projective scheme that is the union of smooth Y_i intersecting along a common smooth divisor $D \cong D_i \subset Y_i$.
- Goal: construct a moduli space for X_0 and a **relative** moduli space (resp. invariant) $M^{\mathsf{rel}}_{Y,D}$ (resp. $i^{\mathsf{rel}}_{Y,D}$) such that
  1. If X is a smoothing of X_0, then $i_X = i_{X_0}$.
  2. i_{X_0} can be written in terms of $i^{\mathsf{rel}}_{Y_1,D_1}$ and $i^{\mathsf{rel}}_{Y_2,D_2}$, i.e. we have a *degeneration formula*.

Setup

- Let $\pi\colon X \to \mathbb{A}^1$ be a *simple degeneration*.

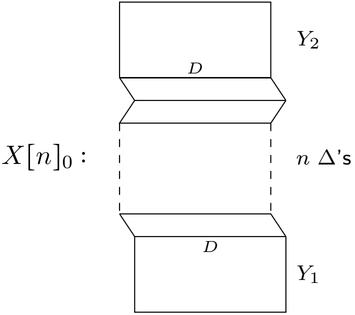


- For $t \neq 0$, the moduli spaces for X_t are well defined.
- The goal is to fill in the central fiber over $0 \in \mathbb{A}^1$ into

$$\coprod_{t \neq 0} M_{X_t}.$$

Central Fiber

- Consider spaces $X[n]_0$, where $X[n]_0$ is created from X_0 as follows:
  - Let $\Delta = \mathbb{P}(N_{D/Y} \oplus \mathcal{O}_D)$.
  - Insert n copies of Δ to $D \subset X_0$.



Central Fiber (cont.)

- Set-theoretically, the central fiber M_{X_0} will be

$$\{\text{stable objects on } X[n]_0 : n \geq 0\} / \sim ,$$

where stable means that the automorphism group is finite and

  - For GW, our objects are pre-stable maps to $X[n]_0$ such that only the nodes can map to the singular divisors of $X[n]_0$.
  - For DT, our objects are closed subschemes of $X[n]_0$ that are normal to the singular divisors of $X[n]_0$.

Main Result for GW and DT Theories

- By filling in the central fiber, the family $\coprod_{t\in\mathbb{A}^1} M_{X_t}$ becomes a proper and separated Deligne-Mumford stack.
- These moduli stacks have virtual fundamental class and thus have enumerative invariants.
- We can also define relative invariants that give rise to a degeneration formula, e.g. for GW invariants, we have for $t \neq 0$,

$$\Psi^{X_t}_{g,n,d}(\alpha(t)) = \sum_{\gamma} \frac{m(\gamma)}{|\operatorname{Aut}(\gamma)|}\left[\Psi^{Y_1^{\text{rel}}}_{\Gamma_1}(j_1^*\alpha(0), b)\cdot \Psi^{Y_2^{\text{rel}}}_{\Gamma_2}(j_2^*\alpha(0), b^*)\right],$$

Related Works

- We can loosen the conditions on our smooth divisor D. Indeed, the degeneration formula works when D is a normal crossings divisor.
  - Chen and Abramovich (2011) & Gross and Siebert (2011) for GW.
  - Maulik and Ranganathan (2020) for DT.
- Other moduli spaces, such as the moduli space of (semi)stable sheaves.
  - Gieseker and Li (1994) created a moduli space of Simpson semistable sheaves for degenerate surfaces, but their construction does not allow a degeneration formula.
  - Kuhn (2023) provided a degeneration formula for fibered surfaces.